

# COMPLETE DECOMPOSITIONS OF COXETER GROUPS ORBIT PRODUCTS OF $A_2$ , $C_2$ , $G_2$ AND $H_2$

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## ABSTRACT.

Orbits of the Weyl reflection groups attached to the simple Lie groups  $A_2$ ,  $C_2$ ,  $G_2$  and Coxeter group  $H_2$  are considered. For each of the groups products of any two orbits are decomposed into the union of the orbits. Results are presented in a single formula for each of the groups.

## 1. INTRODUCTION

Finite reflections groups, called the Coxeter groups, split into two classes: crystallographic (Weyl groups of semisimple Lie algebras) and non-crystallographic groups [1, 2].

There are  $n$  reflections that generate the Coxeter group  $G$  which act in real  $n$  dimensional euclidian space  $\mathbb{R}^n$ . All the distinct points that can be obtained by the action of  $G$  on a selected single point (the ‘seed’ point) of  $\mathbb{R}^n$  form the Coxeter group orbit. Every lattice point belongs to precisely one  $G$ -orbit. The Weyl group orbits are constituents of weight systems of finite dimensional irreducible representations of the algebras. Each weight system is a union of several Weyl group orbits. To determine these constituents for a given representation of a given simple Lie algebra, it is a computational problem for which an efficient algorithm is known [3, 4]. Unlike the weight systems which grow without limits with increasing representations, the  $G$ -orbits are finite in size for a given Lie algebra. Indeed, the largest number of distinct points (‘weights’) an orbit can have, is the order of the Weyl group of the Lie algebra.

The points that form a given orbit are relatively easy to calculate, starting from any one of them, by repeated application of the reflections generating the Coxeter group  $G$ . One almost always chooses as the seed point for such a calculation the unique (‘dominant’) point of the orbit. This point is easily identified because it is the only one in every orbit that has non-negative coordinate in  $\omega$ -basis. A  $G$ -orbit of a Lie algebra of rank  $n$  can be viewed as an  $n$ -dimensional polytope [5] with orbit points as its vertices (0-dimensional faces). Faces of dimensions up to  $(n - 1)$  are also readily described [6].

Tensor products of irreducible representations of the Lie algebras are in one-to-one correspondence with the tensor product of the weight systems. Hence the decomposition of product into irreducible components leads naturally to the problem of decomposition of tensor products of Coxeter group orbits.

Decomposition of products of weight systems into irreducible ones is a familiar problem in representation theory. It is frequently calculated in terms of the products

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of weight systems. Complexity of the decomposition problem rapidly increases with increasing representation that are being multiplied. The only known way how to provide the decomposition of all products of representations for an algebra of rank greater than 1, is by means of the appropriate generating function [7]. Unfortunately such generating function is known only for  $A_2$ . Due to frequent use in particle, nuclear and atomic physics, many special cases of product decompositions are found in the physics literature [8, 9].

Computing separately decomposition of products of Coxeter group orbits allows one to simplify the task of decomposing of the products of the weights systems.

There are two computational task involved when one decomposes the product of weight system:

- (i) Computing the multiplicity of the Coxeter group orbits in any representation involved.
- (ii) Decomposing products of the individual  $G$ -orbits. This is the problem solved in this paper.

The situation is quite different when one considers separately the second problem, namely decomposition of products of  $G$ -orbits into the union of individual orbits. Since the number of points in any orbit cannot exceed the order of the corresponding Coxeter group and it is easily determined in all cases, the decomposition problem for orbits is a finite one, no matter how large the dominant weights may be. Therefore at least, in principle, all the decompositions for a fixed group can be explicitly solved. For the first problem an independent algorithmic solution is known [3, 4, 10]. Its efficiency exceeds all needs that may arise in foreseeable future.

In the paper the second problem, decomposition of products of two orbits, is solved explicitly for the three simple Lie algebras  $A_2$ ,  $C_2$ ,  $G_2$  and non-crystallographic group  $H_2$ . The solution is presented separately for each group and the sketch prove for  $A_2$  is given. Demonstrations of formulae for the others groups one could do analogously to the presented one. For  $A_2$  formula for the decomposition of product of two orbits was presented firstly in [11].

There are other problems where the  $G$ -orbits are essential. Description of reflection generated polytopes which in a simple version (all vertices being in one  $G$ -orbit) is found in [6], symmetries of Clebsch-Gordan coefficients for groups of rank  $\geq 2$  can be formulated in terms of the  $G$ -orbits [12]. The  $G$ -orbits have been used in description of viruses [13].

Most predictable exploitation of Weyl group orbits is in extensive computations with representations of semisimple Lie algebras/groups such as decomposition of tensor products, see one of the largest examples in [14], in branching rules computation, i.e. restriction to representation of subgroups, [15, 16], or for the others [17]. Another exploitation for Coxeter group is in Fourier expansions of digital data on multidimensional lattices. Particularly when a series of similar size orbits will be needed.

Products of two orbit functions can be uniquely decomposed in orbit functions, corresponding to the same Coxeter group, using decomposition of products of two orbits, see [18].

## 2. PRELIMINARIES

The Coxeter groups  $A_2$ ,  $C_2$ ,  $G_2$  and  $H_2$  are groups of order 6, 8, 12 and 10, respectively, generated by reflection in two mirrors intersecting under the angle

$\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$  and  $\frac{\pi}{5}$  at the origin of the real Euclidean space  $\mathbb{R}^2$ . In physics these are the dihedral groups of order 6, 8, 12, 10, respectively.

In order to treat all these cases in uniform way, it is advantageous to work in  $\mathbb{R}^2$  with a pair of dual bases. The  $\alpha$ -basis (simple root basis) is defined by the scalar products

$$\begin{aligned} A_2 : \quad & \langle \alpha_1 | \alpha_1 \rangle = \langle \alpha_2 | \alpha_2 \rangle = 2, \quad \langle \alpha_1 | \alpha_2 \rangle = -1, \\ C_2 : \quad & \langle \alpha_1 | \alpha_1 \rangle = 1, \quad \langle \alpha_2 | \alpha_2 \rangle = 2, \quad \langle \alpha_1 | \alpha_2 \rangle = -1, \\ G_2 : \quad & \langle \alpha_1 | \alpha_1 \rangle = 2, \quad \langle \alpha_2 | \alpha_2 \rangle = \frac{2}{3}, \quad \langle \alpha_1 | \alpha_2 \rangle = -1, \\ H_2 : \quad & \langle \alpha_1 | \alpha_1 \rangle = \langle \alpha_2 | \alpha_2 \rangle = 2, \quad \langle \alpha_1 | \alpha_2 \rangle = -\tau, \quad \text{where } \tau = \frac{1}{2}(1 + \sqrt{5}). \end{aligned}$$

The  $\omega$ -basis is defined as dual to  $\alpha$ -basis,

$$\langle \omega_k | \alpha_j \rangle = \frac{\langle \alpha_j | \alpha_k \rangle}{2} \delta_{jk}. \quad (1)$$

Explicitly,

$$\begin{aligned} A_2 : \quad & \omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \omega_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2, \quad \alpha_1 = 2\omega_1 - \omega_2, \quad \alpha_2 = -\omega_1 + 2\omega_2, \\ C_2 : \quad & \omega_1 = \alpha_1 + \frac{1}{2}\alpha_2, \quad \omega_2 = \alpha_1 + \alpha_2, \quad \alpha_1 = 2\omega_1 - \omega_2, \quad \alpha_2 = -2\omega_1 + 2\omega_2, \\ G_2 : \quad & \omega_1 = 2\alpha_1 + 3\alpha_2, \quad \omega_2 = \alpha_1 + 2\alpha_2, \quad \alpha_1 = 2\omega_1 - 3\omega_2, \quad \alpha_2 = -\omega_1 + 2\omega_2 \end{aligned}$$

and

$$\begin{aligned} H_2 : \quad & \omega_1 = \frac{1}{5}((4 + 2\tau)\alpha_1 + (1 + 3\tau)\alpha_2), \quad \alpha_1 = 2\omega_1 - \tau\omega_2, \\ & \omega_2 = \frac{1}{5}((1 + 3\tau)\alpha_1 + (4 + 2\tau)\alpha_2), \quad \alpha_2 = -\tau\omega_1 + 2\omega_2. \end{aligned} \quad (2)$$

In all cases the reflections  $r_1$  and  $r_2$  that generate  $G$  as follows

$$r_k x = x - \frac{2\langle x | \alpha_k \rangle}{\langle \alpha_k | \alpha_k \rangle} \alpha_k, \quad k = 1, 2, \quad x \in \mathbb{R}^2. \quad (3)$$

In particular,

$$r_k 0 = 0, \quad r_k \omega_j = \omega_j - \delta_{jk} \alpha_k, \quad r_k \alpha_k = -\alpha_k.$$

An orbit of  $G$  is the set of distinct points generated from a seed point  $x \in \mathbb{R}^2$  by repeated action of reflections (3). Such an orbit contains at most as many points as is the order of  $G$ . Each orbit contains precisely one point with non-negative coordinates in  $\omega$ -basis. The orbit  $O(\lambda)$  is specified by this point, called the dominant point.

The orbits could be distinguished according to the position of their points, in particular of the dominant point. It is either point of the weight lattice  $P$  or not in  $\mathbb{R}^2$ .

The set of all dominants weights of  $P$  is in nonnegative sector  $P^+$ , where

$$P^+ = \mathbb{Z}^{\geq 0} \omega_1 + \mathbb{Z}^{\geq 0} \omega_2 \subset P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{R}^2.$$

If  $G$  stands for the non-crystallographic group  $H_2$ , the role of  $\mathbb{Z}$  is played by  $\mathbb{Z}[\tau]$ , quadratic extension of integer numbers by  $\tau$ , see [19, 20].

The size  $|O(\lambda)|$  of an orbit  $O(\lambda)$  is the number of distinct points it contains. In all our cases it takes three values:

$$|O(\lambda)| = \begin{cases} |G| & \text{for } \lambda = (a, b) \\ \frac{1}{2}|G| & \text{for } \lambda = (a, 0) \text{ or } \lambda = (0, b) \\ 1 & \text{for } \lambda = (0, 0) \end{cases},$$

where  $a, b > 0$ .

The product of two orbit functions of  $G$  is the set of points obtained by adding to every point of one orbit every point of the other orbit. Therefore the orbit sizes multiply,  $|O(\lambda) \otimes O(\lambda')| = |O(\lambda)| \cdot |O(\lambda')|$ . The products decompose into the sum of several orbits

$$O(\lambda) \otimes O(\lambda') = O(\lambda') \otimes O(\lambda) = O(\lambda + \lambda') \cup \dots \cup mO(\lambda + \bar{\lambda}'), \quad m \in \mathbb{N}. \quad (4)$$

Here  $\bar{\lambda}$  stands for the lowest weight of the orbit  $O(\lambda)$ . If the sum  $\lambda + \bar{\lambda}'$  is not a dominant weight, it should be reflected into the dominant weight of its orbit.

Except for the rank one of the group  $G$ , there has been no general formula for finding the terms in the decomposition (4). Such formulas are found below for the orbits of the four rank 2 groups.

### 3. DECOMPOSITIONS OF PRODUCTS OF ORBITS OF $A_2$

There is a useful general symmetry property of all orbits  $O(\lambda)$  of  $A_2$ , where  $\lambda = (a, b) = a\omega_1 + b\omega_2$ :

$$O(a, b) = -O(b, a), \quad a, b \in \mathbb{R}. \quad (5)$$

In most cases we are interested in  $a, b \in \mathbb{Z}$ , however (5) is valid for any real  $a$  and  $b$ . For every point  $x \in O(a, b)$  there is the point  $-x \in O(b, a)$ . The property is a consequence of the outer automorphism of  $A_2$ .

Another useful general hierarchy of orbits  $O(a, b)$  of  $A_2$  with integer  $a$  and  $b$ , is their splitting into three mutually exclusive congruence classes according to the value of their congruence number  $K(a, b)$

$$K(a, b) \equiv 2a + b \pmod{3}, \quad a, b \in \mathbb{Z}.$$

All points of an orbit are in the same congruence class. It is the consequence of the fact that difference between two points of the same orbit is an integer linear combination of simple root, and all simple roots are in the same congruence class, namely 0. During the multiplication of orbits, their congruence numbers add up. All orbits in the decomposition belong to that congruence class, see [18, 21].

Let  $a, b \in \mathbb{Z}^{>0}$ . There are four types of orbits  $O(a, b)$ :

$$\begin{aligned} O(0, 0) &= \{(0, 0)\}, \\ O(a, 0) &= \{(a, 0), (-a, a), (0, -a)\}, \quad O(0, b) = \{(0, b), (b, -b), (-b, 0)\}, \\ O(a, b) &= \{(a, b), (-a, a+b), (a+b, -b), (-a-b, a), (b, -a-b), (-b, -a)\}. \end{aligned}$$

For  $\lambda = (a_1, a_2) = a_1\omega_1 + a_2\omega_2$  and  $\lambda' = (b_1, b_2) = b_1\omega_1 + b_2\omega_2$  using duality of bases, see (1), one can get following relations:

$$\begin{aligned} \langle \lambda \mid \alpha_j \rangle &= a_j, \\ \langle \lambda' \mid \alpha_j \rangle &= b_j, \end{aligned} \quad \text{for } j = 1, 2 \text{ and } a_1, a_2, b_1, b_2 \in \mathbb{Z}^{>0} \quad (6)$$

which are useful in the proposition below.

#### Proposition 3.1.

*Decomposition of the product (4) of two orbits of  $A_2$  with dominant weights  $\lambda = (a_1, a_2)$  and  $\lambda' = (b_1, b_2)$  for  $a_1, a_2, b_1, b_2 \in \mathbb{Z}^{>0}$  is given by the following formula*

$$\begin{aligned}
& O(\lambda) \otimes O(\lambda') \\
&= k_1 O(\lambda + \lambda') \\
&\cup k_2 O \left( \left| \langle \lambda - \lambda' | \alpha_1 \rangle \right|, \left| \langle \lambda + \lambda' | \frac{1}{2}\alpha_1 + \alpha_2 \rangle - \frac{1}{2} \left| \langle \lambda - \lambda' | \alpha_1 \rangle \right| \right| \right) \\
&\cup k_3 O \left( \left| \langle \lambda + \lambda' | \alpha_1 + \frac{1}{2}\alpha_2 \rangle - \frac{1}{2} \left| \langle \lambda - \lambda' | \alpha_2 \rangle \right| \right|, \left| \langle \lambda - \lambda' | \alpha_2 \rangle \right| \right) \\
&\cup k_4 O \left( \left| \langle \lambda' | \alpha_1 + \frac{1}{2}\alpha_2 \rangle + \frac{1}{2} \langle \lambda | \alpha_2 - \alpha_1 \rangle - \frac{1}{2} \left| \langle \lambda - \lambda' | \alpha_2 \rangle + \langle \lambda | \alpha_1 \rangle \right| \right|, \right. \\
&\quad \left. \left| \langle \lambda | \frac{1}{2}\alpha_1 + \alpha_2 \rangle + \frac{1}{2} \langle \lambda' | \alpha_1 - \alpha_2 \rangle - \frac{1}{2} \left| \langle \lambda' - \lambda | \alpha_1 \rangle + \langle \lambda' | \alpha_2 \rangle \right| \right| \right) \\
&\cup k_5 O \left( \left| \langle \lambda | \alpha_1 + \frac{1}{2}\alpha_2 \rangle + \frac{1}{2} \langle \lambda' | \alpha_2 - \alpha_1 \rangle - \frac{1}{2} \left| \langle \lambda' - \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle \right| \right|, \right. \\
&\quad \left. \left| \langle \lambda' | \frac{1}{2}\alpha_1 + \alpha_2 \rangle + \frac{1}{2} \langle \lambda | \alpha_1 - \alpha_2 \rangle - \frac{1}{2} \left| \langle \lambda - \lambda' | \alpha_1 \rangle + \langle \lambda | \alpha_2 \rangle \right| \right| \right) \\
&\cup k_6 O \left( \left| \left| \langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle \right| - \left| \min \{ \langle \lambda | \alpha_1 \rangle - \langle \lambda' | \alpha_2 \rangle, \langle \lambda' | \alpha_1 \rangle - \langle \lambda | \alpha_2 \rangle, 0 \} \right| \right|, \right. \\
&\quad \left. \left| \left| \langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle \right| - \left| \min \{ -\langle \lambda | \alpha_1 \rangle + \langle \lambda' | \alpha_2 \rangle, -\langle \lambda' | \alpha_1 \rangle + \langle \lambda | \alpha_2 \rangle, 0 \} \right| \right| \right)
\end{aligned} \tag{7}$$

where the multiplicities  $k_1, \dots, k_6$  are given in terms of orbits sizes by:

$$\begin{aligned}
k_1 &= \frac{1}{6} \frac{|O(\lambda)||O(\lambda')|}{|O(\lambda+\lambda')|} \\
k_2 &= \frac{1}{6} \frac{|O(\lambda)||O(\lambda')|}{\left| O \left( \left| \langle \lambda - \lambda' | \alpha_1 \rangle \right|, \left| \langle \lambda + \lambda' | \frac{1}{2}\alpha_1 + \alpha_2 \rangle - \frac{1}{2} \left| \langle \lambda - \lambda' | \alpha_1 \rangle \right| \right| \right) \right|} \\
k_3 &= \frac{1}{6} \frac{|O(\lambda)||O(\lambda')|}{\left| O \left( \langle \lambda + \lambda' | \alpha_1 + \frac{1}{2}\alpha_2 \rangle - \frac{1}{2} \left| \langle \lambda - \lambda' | \alpha_2 \rangle \right|, \left| \langle \lambda - \lambda' | \alpha_2 \rangle \right| \right) \right|} \\
k_4 &= \frac{1}{6} |O(\lambda)||O(\lambda')| / \left| O \left( \left| \langle \lambda' | \alpha_1 + \frac{1}{2}\alpha_2 \rangle + \frac{1}{2} \langle \lambda | \alpha_2 - \alpha_1 \rangle - \frac{1}{2} \left| \langle \lambda - \lambda' | \alpha_2 \rangle + \langle \lambda | \alpha_1 \rangle \right| \right|, \right. \right. \\
&\quad \left. \left. \left| \langle \lambda | \frac{1}{2}\alpha_1 + \alpha_2 \rangle + \frac{1}{2} \langle \lambda' | \alpha_1 - \alpha_2 \rangle - \frac{1}{2} \left| \langle \lambda' - \lambda | \alpha_1 \rangle + \langle \lambda' | \alpha_2 \rangle \right| \right| \right) \right| \\
k_5 &= \frac{1}{6} |O(\lambda)||O(\lambda')| / \left| O \left( \left| \langle \lambda | \alpha_1 + \frac{1}{2}\alpha_2 \rangle + \frac{1}{2} \langle \lambda' | \alpha_2 - \alpha_1 \rangle - \frac{1}{2} \left| \langle \lambda' - \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle \right| \right|, \right. \right. \\
&\quad \left. \left. \left| \langle \lambda' | \frac{1}{2}\alpha_1 + \alpha_2 \rangle + \frac{1}{2} \langle \lambda | \alpha_1 - \alpha_2 \rangle - \frac{1}{2} \left| \langle \lambda - \lambda' | \alpha_1 \rangle + \langle \lambda | \alpha_2 \rangle \right| \right| \right) \right| \\
k_6 &= \frac{1}{6} |O(\lambda)||O(\lambda')| \cdot \\
&\quad 1 / \left| O \left( \left| \langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle \right| - \left| \min \{ \langle \lambda | \alpha_1 \rangle - \langle \lambda' | \alpha_2 \rangle, \langle \lambda' | \alpha_1 \rangle - \langle \lambda | \alpha_2 \rangle, 0 \} \right| \right|, \right. \\
&\quad \left. \left| \langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle \right| - \left| \min \{ -\langle \lambda | \alpha_1 \rangle + \langle \lambda' | \alpha_2 \rangle, -\langle \lambda' | \alpha_1 \rangle + \langle \lambda | \alpha_2 \rangle, 0 \} \right| \right| \right|.
\end{aligned}$$

Sketch of the prove is given in the Appendix A.

Note that the congruences class of each term in the decomposition (7) coincide. Namely the congruence class which is the sum of congruence classes of factors of  $O(a_1, a_2) \otimes O(b_1, b_2)$ . Lets illustrate this fact in an example.

### Example 1.

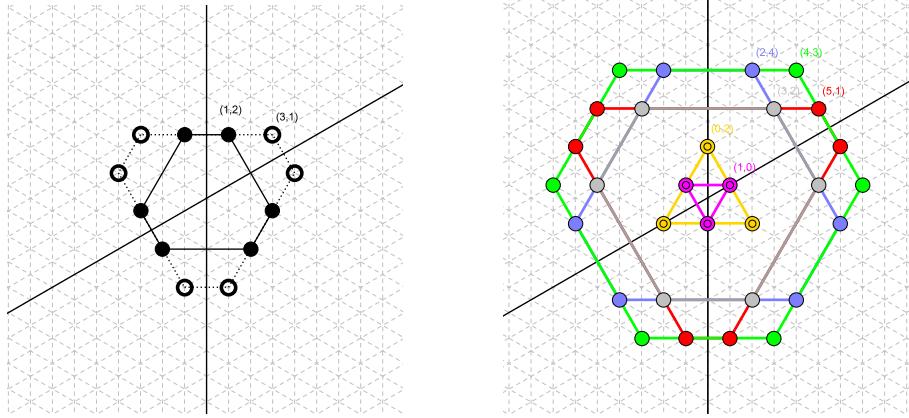
$$\begin{aligned}
O(1, 2) \otimes O(3, 1) &= O(4, 3) \cup O(2, 4) \\
&\cup O(3, 2) \cup O(5, 1) \cup 2O(0, 2) \cup 2O(1, 0),
\end{aligned} \tag{8}$$

then one has

$$K(1, 2) + K(3, 1) = 2 \pmod{3},$$

$$K(4,3) = K(0,2) = K(2,4) = K(3,2) = K(5,1) = K(1,0) = 2 \pmod{3}.$$

Orbits and decomposition of product (8) are presented in figure 1.



(a) The six points of the orbit  $O(1,2)$  and of  $O(3,1)$  of  $A_2$ . The straight lines are the reflection mirrors containing  $\omega_1$  and  $\omega_2$ . The dominant points are indicated in the positive sector. Dashed lines are the directions of the weight lattice axes of  $A_2$ .

(b) The six orbits of the decomposition of the product  $O(1,2) \otimes O(3,1)$  of  $A_2$ . Four of them are hexagons and two are triangles taken twice what is denoted by double dots. The straight lines are the reflection mirrors containing  $\omega_1$  and  $\omega_2$ . The dominant points are indicated in the positive sector. Dashed lines are the directions of the weight lattice axes of  $A_2$ .

FIGURE 1.

#### 4. DECOMPOSITIONS OF PRODUCTS OF ORBITS OF $C_2$

A useful general symmetry property of all orbits of  $C_2$ :

$$x \in O(a,b) \iff -x \in O(a,b), \quad a, b \in \mathbb{R}^n. \quad (9)$$

In most cases we are interested in  $a, b \in \mathbb{Z}$ , however (9) is valid for any real  $a$  and  $b$ . The points  $\pm x \in O(a,b)$ .

A useful general hierarchy of orbits  $O(a,b)$  of  $C_2$  with integer  $a$  and  $b$ , is their splitting into two mutually exclusive congruence classes according to the value of their congruence number  $K(a,b)$

$$K(a,b) = a \pmod{2}, \quad a, b \in \mathbb{Z}.$$

All points of an orbit are in the same congruence class. It is the consequence of the fact that difference between two points of the same orbit is an integer linear combination of simple root, and all simple roots are in the congruence class 0. During the multiplication of orbits, their congruence numbers add up. All orbits in the decomposition belong to that congruence class.

There are three kinds of nontrivial orbits for this case and could be calculated very easily. One can find it in for example [18, 21]:

$$\begin{aligned}
O(0,0) &= \{(0,0)\}, \\
O(a,0) &= \{\pm(a,0), \pm(-a,a)\}, \\
O(0,b) &= \{\pm(0,b), \pm(2b,-b)\}, \\
O(a,b) &= \{\pm(a,b), \pm(-a,a+b), \pm(a+2b,-b), \pm(a+2b,-a-b)\}.
\end{aligned}$$

By analogy to (6) one has  $\lambda = (a_1, a_2) = a_1\omega_1 + a_2\omega_2$  and  $\lambda' = (b_1, b_2) = b_1\omega_1 + b_2\omega_2$  using duality of bases one can get following relations:

$$\begin{aligned}
\langle \lambda | \alpha_1 \rangle &= \frac{1}{2}a_1, & \langle \lambda' | \alpha_1 \rangle &= \frac{1}{2}b_1, & \text{for } a_1, a_2, b_1, b_2 \in \mathbb{Z}^{\geq 0} \\
\langle \lambda | \alpha_2 \rangle &= a_2, & \langle \lambda' | \alpha_2 \rangle &= b_2,
\end{aligned} \tag{10}$$

which are useful in the proposition below.

#### Proposition 4.1.

*Decomposition of the product (4) of two orbits of  $C_2$  with dominant weights  $\lambda = (a_1, a_2)$  and  $\lambda' = (b_1, b_2)$  for  $a_1, a_2, b_1, b_2 \in \mathbb{Z}^{\geq 0}$  is given by the following formula*

$$\begin{aligned}
& O(\lambda) \otimes O(\lambda') \\
&= k_1 O(\lambda + \lambda') \\
&\cup k_2 O(|\langle \lambda - \lambda' | 2\alpha_1 \rangle|, |\langle \lambda + \lambda' | \alpha_1 + \alpha_2 \rangle - |\langle \lambda - \lambda' | \alpha_1 \rangle|) \\
&\cup k_3 O(|\langle \lambda + \lambda' | 2\alpha_1 + \alpha_2 \rangle - |\langle \lambda - \lambda' | \alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_2 \rangle|) \\
&\cup k_4 O(|\langle \lambda + \lambda' | \alpha_2 \rangle + \langle \lambda | 2\alpha_1 \rangle - |\langle \lambda' | 2\alpha_1 \rangle + \langle \lambda' - \lambda | \alpha_2 \rangle|, \\
&\quad |\langle \lambda + \lambda' | \alpha_1 \rangle + \langle \lambda' | \alpha_2 \rangle - |\langle \lambda' - \lambda | \alpha_1 \rangle - \langle \lambda | \alpha_2 \rangle| |) \\
&\cup k_5 O(\min\{|\langle 2\lambda | \alpha_1 + \alpha_2 \rangle + \langle \lambda' | 2\alpha_1 \rangle|, |\langle \lambda | 2\alpha_1 \rangle - \langle 2\lambda' | \alpha_1 + \alpha_2 \rangle|\}, \\
&\quad |\langle \lambda + \lambda' | \alpha_1 \rangle + \langle \lambda | \alpha_2 \rangle - |\langle \lambda - \lambda' | \alpha_1 \rangle - \langle \lambda' | \alpha_2 \rangle| |) \tag{11} \\
&\cup k_6 O(2 \min\{|\langle \lambda - \lambda' | \alpha_1 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle|\}, \\
&\quad \min\{|\langle \lambda - \lambda' | \alpha_2 \rangle|, |\langle \lambda - \lambda' | 2\alpha_1 + \alpha_2 \rangle|\}) \\
&\cup k_7 O(2 \min\{|\langle \lambda + \lambda' | \alpha_1 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle|\}, \\
&\quad |\langle \lambda + \lambda' | \alpha_1 \rangle - |\langle \lambda' - \lambda | \alpha_1 + \alpha_2 \rangle| |) \\
&\cup k_8 O(|\langle \lambda + \lambda' | \alpha_2 \rangle - |\langle \lambda' - \lambda | 2\alpha_1 + \alpha_2 \rangle|, \\
&\quad \min\{|\langle \lambda + \lambda' | \alpha_2 \rangle|, |\langle \lambda - \lambda' | 2\alpha_1 + \alpha_2 \rangle|\})
\end{aligned}$$

where the multiplicities  $k_1, \dots, k_8$  are given in terms of orbits sizes by:

$$\begin{aligned}
k_1 &= \frac{1}{8} \frac{|O(\lambda)| |O(\lambda')|}{|O(\lambda + \lambda')|} \\
k_2 &= \frac{1}{8} \frac{|O(\lambda)| |O(\lambda')|}{|O(|\langle \lambda - \lambda' | 2\alpha_1 \rangle|, |\langle \lambda + \lambda' | \alpha_1 + \alpha_2 \rangle - |\langle \lambda - \lambda' | \alpha_1 \rangle|)|} \\
k_3 &= \frac{1}{8} \frac{|O(\lambda)| |O(\lambda')|}{|O(|\langle \lambda + \lambda' | 2\alpha_1 + \alpha_2 \rangle - |\langle \lambda - \lambda' | \alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_2 \rangle|)|} \\
k_4 &= \frac{1}{8} |O(\lambda)| |O(\lambda')| \left/ \left| O(|\langle \lambda + \lambda' | \alpha_2 \rangle + \langle \lambda | 2\alpha_1 \rangle - |\langle \lambda' | 2\alpha_1 \rangle + \langle \lambda' - \lambda | \alpha_2 \rangle| |, \right. \right. \\
&\quad \left. \left. |\langle \lambda + \lambda' | \alpha_1 \rangle + \langle \lambda' | \alpha_2 \rangle - |\langle \lambda' - \lambda | \alpha_1 \rangle - \langle \lambda | \alpha_2 \rangle| | \right) \right| \\
k_5 &= \frac{1}{8} |O(\lambda)| |O(\lambda')| \left/ \left| O(\min\{|\langle 2\lambda | \alpha_1 + \alpha_2 \rangle + \langle \lambda' | 2\alpha_1 \rangle|, |\langle \lambda | 2\alpha_1 \rangle - \langle 2\lambda' | \alpha_1 + \alpha_2 \rangle|\}, \right. \right. \\
&\quad \left. \left. |\langle \lambda + \lambda' | \alpha_1 \rangle + \langle \lambda | \alpha_2 \rangle - |\langle \lambda - \lambda' | \alpha_1 \rangle - \langle \lambda' | \alpha_2 \rangle| | \right) \right|
\end{aligned}$$

$$\begin{aligned}
k_6 &= \frac{1}{8} |O(\lambda)| |O(\lambda')| \\
k_7 &= \frac{1}{8} \frac{|O(2 \min\{|\langle \lambda - \lambda' | \alpha_1 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle|\}, \min\{|\langle \lambda - \lambda' | \alpha_2 \rangle|, |\langle \lambda - \lambda' | 2\alpha_1 + \alpha_2 \rangle|\})|}{|O(\lambda)| |O(\lambda')|} \\
k_8 &= \frac{1}{8} \frac{|O(\lambda + \lambda' | \alpha_1), |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle|, |\langle \lambda + \lambda' | \alpha_1\rangle - |\langle \lambda' - \lambda | \alpha_1 + \alpha_2 \rangle||)}{|O(\lambda)| |O(\lambda')|} \\
&\quad \cdot |O(|\langle \lambda + \lambda' | \alpha_2 \rangle - |\langle \lambda' - \lambda | 2\alpha_1 + \alpha_2 \rangle|, \min\{|\langle \lambda + \lambda' | \alpha_2 \rangle, |\langle \lambda - \lambda' | 2\alpha_1 + \alpha_2 \rangle|\})|
\end{aligned}$$

The formula (11) can be verified in the same way as (7).

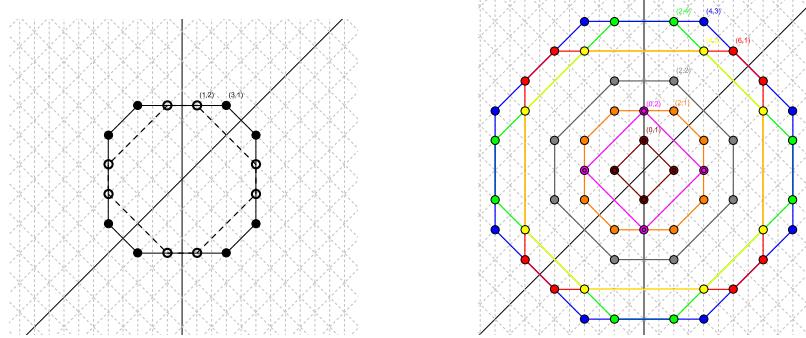
Analogously sum of congruence classes of product's factors is equal the congruence class of each term in the decomposition (11), which is easy to verify by straightforward calculation of congruence classes. Let illustrate this fact in an example.

### Example 2.

$$O(1, 2) \otimes O(3, 1)$$

$$= O(4, 3) \cup O(6, 1) \cup O(4, 2) \cup O(2, 4) \cup O(2, 2) \cup O(2, 1) \cup 2O(0, 2) \cup 2O(0, 1) \quad (12)$$

Orbits and decomposition of product (12) are presented in figure 2. More advanced examples one could find in appendix B.



(a) The eight points of the orbit  $O(1, 2)$  and of  $O(3, 1)$  of  $C_2$ . The straight lines are the reflection mirrors containing  $\omega_1$  and  $\omega_2$ . The dominant points are indicated in the positive sector. Dashed lines are the directions of the weight lattice axes of  $C_2$ .

(b) The eight orbits of the decomposition of the product  $O(1, 2) \otimes O(3, 1)$  of  $C_2$ . Six of them are octagons and two are squares taken twice what is denoted by double dots. The straight lines are the reflection mirrors containing  $\omega_1$  and  $\omega_2$ . The dominant points are indicated in the positive sector. Dashed lines are the directions of the weight lattice axes of  $C_2$ .

FIGURE 2.

## 5. DECOMPOSITIONS OF PRODUCTS OF ORBITS OF $G_2$

A useful general symmetry property of all orbits of  $G_2$ :

$$x \in O(a, b) \iff -x \in O(a, b), \quad a, b \in \mathbb{R}^n. \quad (13)$$

In most cases we are interested in  $a, b \in \mathbb{Z}$ , however (13) is valid for any real  $a$  and  $b$ . The points  $\pm x \in O(a, b)$ . All weights are in the same congruence class.

For this group one has also three kinds of nontrivial orbits for this case and could be calculated very easily. One can find it in for example [18, 21]:

$$\begin{aligned} O(0, 0) &= \{(0, 0)\}, \\ O(a, 0) &= \{\pm(a, 0), \pm(-a, 3a), \pm(2a, -3a)\}, \\ O(0, b) &= \{\pm(0, b), \pm(b, -b), \pm(-b, 2b)\}, \\ O(a, b) &= \{\pm(a, b), \pm(-a, 3a + b), \pm(a + b, -b), \\ &\quad \pm(2a + b, -3a - b), \pm(-a - b, 3a + 2b), \pm(-2a - b, 3a + 2b)\}. \end{aligned}$$

As before product of two orbits of  $G_2$  is written in terms of  $\lambda = (a_1, a_2) = a_1\omega_1 + a_2\omega_2$  and  $\lambda' = (b_1, b_2) = b_1\omega_1 + b_2\omega_2$  and simple roots, i.e.:

$$\begin{aligned} \langle \lambda | \alpha_1 \rangle &= a_1, & \langle \lambda' | \alpha_1 \rangle &= b_1, \\ \langle \lambda | \alpha_2 \rangle &= \frac{1}{3}a_2, & \langle \lambda' | \alpha_2 \rangle &= \frac{1}{3}b_2, \end{aligned} \quad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{Z}^{\geq 0}.$$

### Proposition 5.1.

*Decomposition of the product (4) of two orbits of  $G_2$  with dominant weights  $\lambda = (a_1, a_2)$  and  $\lambda' = (b_1, b_2)$  for  $a_1, a_2, b_1, b_2 \in \mathbb{Z}^{\geq 0}$  is given by the following formula*

$$\begin{aligned} O(\lambda) \otimes O(\lambda') &= k_1 O(\lambda + \lambda') & (14) \\ &= k_1 O(\lambda + \lambda') \\ &\cup k_2 O(|\langle \lambda - \lambda' | \alpha_1 \rangle|, \frac{3}{2}(|\langle \lambda + \lambda' | \alpha_1 + 2\alpha_2 \rangle| - |\langle \lambda - \lambda' | \alpha_1 \rangle|)) \\ &\cup k_3 O(|\langle \lambda + \lambda' | \alpha_1 + \frac{3}{2}\alpha_2 \rangle| - \frac{3}{2}|\langle \lambda - \lambda' | \alpha_2 \rangle|, 3|\langle \lambda - \lambda' | \alpha_2 \rangle|) \\ &\cup k_4 O(\min\{|\langle \lambda | 2\alpha_1 + 3\alpha_2 \rangle| + |\langle \lambda' | \alpha_1 \rangle|, |\langle \lambda - \lambda' | \alpha_1 \rangle| - 3|\langle \lambda' | \alpha_2 \rangle|\}, \\ &\quad 3\min\{|\langle \lambda | \alpha_2 \rangle| + |\langle \lambda' | \alpha_1 + 2\alpha_2 \rangle|, |\langle \lambda | \alpha_1 + \alpha_2 \rangle| - |\langle \lambda' | \alpha_2 \rangle|\}) \\ &\cup k_5 O(\min\{|\langle \lambda | \alpha_1 \rangle| + |\langle \lambda' | 2\alpha_1 + 3\alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_1 \rangle| + 3|\langle \lambda | \alpha_2 \rangle|\}, \\ &\quad 3\min\{|\langle \lambda | \alpha_1 + 2\alpha_2 \rangle| + |\langle \lambda' | \alpha_2 \rangle|, |\langle \lambda | \alpha_2 \rangle| - |\langle \lambda' | \alpha_1 + \alpha_2 \rangle|\}) \\ &\cup k_6 O(\min\{|\langle \lambda | 2\alpha_1 + 3\alpha_2 \rangle| - |\langle \lambda' | \alpha_1 \rangle|, |\langle \lambda | \alpha_1 \rangle| - |\langle \lambda' | 2\alpha_1 + 3\alpha_2 \rangle|\}, \\ &\quad 3\min\{|\langle \lambda | \alpha_1 + 2\alpha_2 \rangle| + |\langle \lambda' | \alpha_2 \rangle|, \\ &\quad |\langle \lambda | \alpha_2 \rangle| + |\langle \lambda' | \alpha_1 + 2\alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle|\}) \\ &\cup k_7 O(\min\{|\langle \lambda - \lambda' | \alpha_1 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + 3\alpha_2 \rangle|, |\langle \lambda - \lambda' | 2\alpha_1 + 3\alpha_2 \rangle|\}, \\ &\quad 3\min\{|\langle \lambda - \lambda' | \alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + 2\alpha_2 \rangle|\}) \\ &\cup k_8 O(\min\{|\langle \lambda + \lambda' | \alpha_1 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + 3\alpha_2 \rangle| + |\langle \lambda | \alpha_1 \rangle|, \\ &\quad |\langle \lambda - \lambda' | \alpha_1 + 3\alpha_2 \rangle| - |\langle \lambda' | \alpha_1 \rangle|, 3\min\{|\langle \lambda | \alpha_1 + \alpha_2 \rangle| - |\langle \lambda' | \alpha_2 \rangle|, \\ &\quad |\langle \lambda - \lambda' | \alpha_1 + 2\alpha_2 \rangle|, |\langle \lambda | \alpha_2 \rangle| - |\langle \lambda' | \alpha_1 + \alpha_2 \rangle|\}) \\ &\cup k_9 O(\min\{|\langle \lambda - \lambda' | \alpha_1 \rangle| + 3|\langle \lambda | \alpha_2 \rangle|, |\langle \lambda - \lambda' | 2\alpha_1 + 3\alpha_2 \rangle|, \\ &\quad |\langle \lambda - \lambda' | \alpha_1 \rangle| - 3|\langle \lambda' | \alpha_2 \rangle|, 3\min\{|\langle \lambda + \lambda' | \alpha_2 \rangle|, \\ &\quad |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle| + |\langle \lambda | \alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle| - |\langle \lambda' | \alpha_2 \rangle|\}) \\ &\cup k_{10} O(\min\{|\langle \lambda + \lambda' | \alpha_1 \rangle| + 3|\langle \lambda' | \alpha_2 \rangle|, |\langle \lambda | 2\alpha_1 + 3\alpha_2 \rangle| - |\langle \lambda' | \alpha_1 \rangle|, \\ &\quad |\langle \lambda - \lambda' | \alpha_1 + 3\alpha_2 \rangle| - |\langle \lambda' | \alpha_1 \rangle|, 3\min\{|\langle \lambda | \alpha_1 + \alpha_2 \rangle| + |\langle \lambda' | \alpha_2 \rangle|, \\ &\quad |\langle \lambda - \lambda' | \alpha_1 + 2\alpha_2 \rangle| + |\langle \lambda' | \alpha_2 \rangle|, |\langle \lambda | \alpha_2 \rangle| - |\langle \lambda' | \alpha_1 + 2\alpha_2 \rangle|\}) \end{aligned}$$

$$\begin{aligned}
& \cup k_{11} O(\min\{\langle \lambda | \alpha_1 + 3\alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle, |\langle \lambda - \lambda' | 2\alpha_1 + 3\alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle|, \\
& \quad |\langle \lambda | \alpha_1 \rangle - \langle \lambda' | 2\alpha_1 + 3\alpha_2 \rangle|\}, 3 \min\{\langle \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 + \alpha_2 \rangle, \\
& \quad |\langle \lambda - \lambda' | \alpha_1 + 2\alpha_2 \rangle - \langle \lambda | \alpha_2 \rangle|, |\langle \lambda | \alpha_1 + 2\alpha_2 \rangle - \langle \lambda' | \alpha_2 \rangle|\}) \\
& \cup k_{12} O(\min\{\langle \lambda | 2\alpha_1 + 3\alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle, |\langle \lambda - \lambda' | \alpha_1 + 3\alpha_2 \rangle|, \\
& \quad \langle \lambda | \alpha_1 \rangle + \langle \lambda' | 2\alpha_1 + 3\alpha_2 \rangle\}, 3 \min\{|\langle \lambda | \alpha_1 + 2\alpha_2 \rangle - \langle \lambda' | \alpha_2 \rangle|, \\
& \quad |\langle \lambda - \lambda' | \alpha_2 \rangle - \langle \lambda' | \alpha_1 + \alpha_2 \rangle|\})
\end{aligned}$$

where the multiplicities  $k_1, \dots, k_{12}$  are given in terms of orbits sizes by:

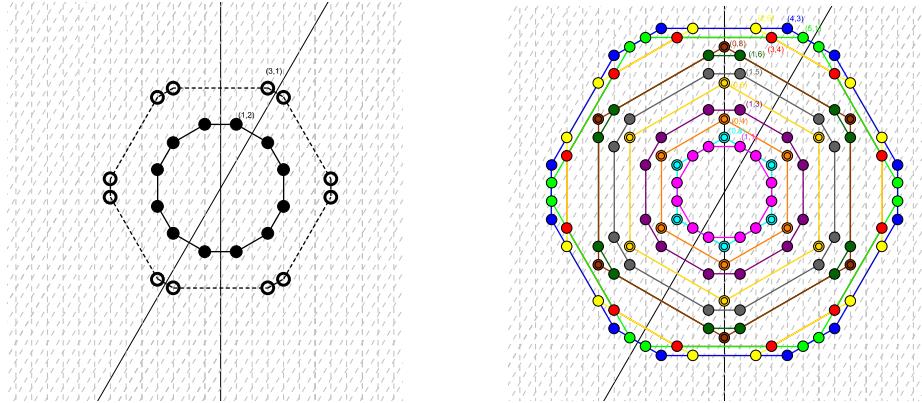
$$\begin{aligned}
k_1 &= \frac{1}{12} \frac{|O(\lambda)||O(\lambda')|}{|O(\lambda + \lambda')|} \\
k_2 &= \frac{1}{12} \frac{|O(\lambda)||O(\lambda')|}{|O(|\langle \lambda - \lambda' | \alpha_1 \rangle|, \frac{3}{2}(\langle \lambda + \lambda' | \alpha_1 + 2\alpha_2 \rangle - |\langle \lambda - \lambda' | \alpha_1 \rangle))|} \\
k_3 &= \frac{1}{12} \frac{|O(\lambda)||O(\lambda')|}{|O(\langle \lambda + \lambda' | \alpha_1 + \frac{3}{2}\alpha_2 \rangle - \frac{3}{2}|\langle \lambda - \lambda' | \alpha_2 \rangle|, 3|\langle \lambda - \lambda' | \alpha_2 \rangle)|} \\
k_4 &= \frac{1}{12} |O(\lambda)||O(\lambda')| / |O(\min\{\langle \lambda | 2\alpha_1 + 3\alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle, |\langle \lambda - \lambda' | \alpha_1 \rangle - 3\langle \lambda' | \alpha_2 \rangle|\}, \\
&\quad 3 \min\{|\langle \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 + 2\alpha_2 \rangle, |\langle \lambda | \alpha_1 + \alpha_2 \rangle - \langle \lambda' | \alpha_2 \rangle|\})| \\
k_5 &= \frac{1}{12} |O(\lambda)||O(\lambda')| / |O(\min\{\langle \lambda | \alpha_1 \rangle + \langle \lambda' | 2\alpha_1 + 3\alpha_2 \rangle, |\langle \lambda - \lambda' | \alpha_1 \rangle + 3\langle \lambda | \alpha_2 \rangle|\}, \\
&\quad 3 \min\{|\langle \lambda | \alpha_1 + 2\alpha_2 \rangle + \langle \lambda' | \alpha_2 \rangle, |\langle \lambda | \alpha_2 \rangle - \langle \lambda' | \alpha_1 + \alpha_2 \rangle|\})| \\
k_6 &= \frac{1}{12} |O(\lambda)||O(\lambda')| / |O(\min\{|\langle \lambda | 2\alpha_1 + 3\alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle|, |\langle \lambda | \alpha_1 \rangle - \langle \lambda' | 2\alpha_1 + 3\alpha_2 \rangle|\}, \\
&\quad 3 \min\{|\langle \lambda | \alpha_1 + 2\alpha_2 \rangle + \langle \lambda' | \alpha_2 \rangle, \langle \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 + 2\alpha_2 \rangle, |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle|\})| \\
k_7 &= \frac{1}{12} |O(\lambda)||O(\lambda')| / |O(\min\{|\langle \lambda - \lambda' | \alpha_1 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + 3\alpha_2 \rangle|, |\langle \lambda - \lambda' | 2\alpha_1 + 3\alpha_2 \rangle|\}, \\
&\quad 3 \min\{|\langle \lambda - \lambda' | \alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + 2\alpha_2 \rangle|\})| \\
k_8 &= \frac{1}{12} |O(\lambda)||\lambda'| / |O(\min\{|\langle \lambda + \lambda' | \alpha_1 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + 3\alpha_2 \rangle + \langle \lambda | \alpha_1 \rangle|, \\
&\quad |\langle \lambda - \lambda' | \alpha_1 + 3\alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle|, 3 \min\{|\langle \lambda | \alpha_1 + \alpha_2 \rangle - \langle \lambda' | \alpha_2 \rangle|, \\
&\quad |\langle \lambda - \lambda' | \alpha_1 + 2\alpha_2 \rangle|, |\langle \lambda | \alpha_2 \rangle - \langle \lambda' | \alpha_1 + \alpha_2 \rangle|\})| \\
k_9 &= \frac{1}{12} |O(\lambda)||O(\lambda')| / |O(\min\{|\langle \lambda - \lambda' | \alpha_1 \rangle + 3\langle \lambda | \alpha_2 \rangle|, |\langle \lambda - \lambda' | 2\alpha_1 + 3\alpha_2 \rangle|, \\
&\quad |\langle \lambda - \lambda' | \alpha_1 \rangle - 3\langle \lambda' | \alpha_2 \rangle|, 3 \min\{|\langle \lambda + \lambda' | \alpha_2 \rangle|, \\
&\quad |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle + \langle \lambda | \alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle - \langle \lambda' | \alpha_2 \rangle|\})| \\
k_{10} &= \frac{1}{12} |O(\lambda)||O(\lambda')| / |O(\min\{|\langle \lambda + \lambda' | \alpha_1 \rangle + 3\langle \lambda' | \alpha_2 \rangle|, |\langle \lambda | 2\alpha_1 + 3\alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle|, \\
&\quad |\langle \lambda - \lambda' | \alpha_1 + 3\alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle|, 3 \min\{|\langle \lambda | \alpha_1 + \alpha_2 \rangle + \langle \lambda' | \alpha_2 \rangle|, \\
&\quad |\langle \lambda - \lambda' | \alpha_1 + 2\alpha_2 \rangle + \langle \lambda' | \alpha_2 \rangle|, |\langle \lambda | \alpha_2 \rangle - \langle \lambda' | \alpha_1 + 2\alpha_2 \rangle|\})| \\
k_{11} &= \frac{1}{12} |O(\lambda)||O(\lambda')| / |O(\min\{|\langle \lambda | \alpha_1 + 3\alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle|, |\langle \lambda - \lambda' | 2\alpha_1 + 3\alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle|, \\
&\quad |\langle \lambda | \alpha_1 \rangle - \langle \lambda' | 2\alpha_1 + 3\alpha_2 \rangle|, 3 \min\{|\langle \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 + \alpha_2 \rangle|, \\
&\quad |\langle \lambda - \lambda' | \alpha_1 + 2\alpha_2 \rangle - \langle \lambda | \alpha_2 \rangle|, |\langle \lambda | \alpha_1 + 2\alpha_2 \rangle - \langle \lambda' | \alpha_2 \rangle|\})| \\
k_{12} &= \frac{1}{12} |O(\lambda)||O(\lambda')| / |O(\min\{|\langle \lambda | 2\alpha_1 + 3\alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle|, |\langle \lambda - \lambda' | \alpha_1 + 3\alpha_2 \rangle|, \\
&\quad |\langle \lambda | \alpha_1 \rangle + \langle \lambda' | 2\alpha_1 + 3\alpha_2 \rangle|, 3 \min\{|\langle \lambda | \alpha_1 + 2\alpha_2 \rangle - \langle \lambda' | \alpha_2 \rangle|, \\
&\quad |\langle \lambda - \lambda' | \alpha_2 \rangle - \langle \lambda' | \alpha_1 + \alpha_2 \rangle|\})|
\end{aligned}$$

The prove of (14) is again done by writing of existing list of all special cases following from it. Then each case of the list is easily verify.

**Example 3.**

$$\begin{aligned}
O(1, 2) \otimes O(3, 1) &= O(4, 3) \cup O(5, 1) \cup O(3, 4) \cup O(2, 6) \cup O(1, 6) \\
&\cup O(1, 5) \cup O(1, 3) \cup 2O(0, 8) \cup 2O(0, 6) \cup 2O(0, 4) \cup 2O(0, 3) \cup O(1, 1),
\end{aligned} \tag{15}$$

Orbits and decomposition of product (15) are presented in figure 3. More advanced examples one could find in appendix C.



(a) The twelve points of the orbit  $O(1, 2)$  and of  $O(3, 1)$  of  $G_2$ . The straight lines are the reflection mirrors containing  $\omega_1$  and  $\omega_2$ . The dominant points are indicated in the positive sector. Dashed lines are the directions of the weight lattice axes of  $G_2$ .

(b) The twelve orbits of the decomposition of the product  $O(1, 2) \otimes O(3, 1)$  of  $G_2$ . Eight of them are dodecagons and four are hexagons taken twice what is denoted by double dots. The straight lines are the reflection mirrors containing  $\omega_1$  and  $\omega_2$ . The dominant points are indicated in the positive sector. Dashed lines are the directions of the weight lattice axes of  $G_2$ .

FIGURE 3.

## 6. DECOMPOSITIONS OF PRODUCTS OF ORBITS OF $H_2$

Some information about Coxeter group  $H_2$  are presented in Appendix D. A useful general symmetry property of all orbits of  $H_2$ :

$$x \in O(a, b) \iff -x \in O(b, a), \quad a, b \in \mathbb{R}^n. \tag{16}$$

In most cases we are interested in  $a, b \in \mathbb{Z}$ , however (16) is valid for any real  $a$  and  $b$ . skip

For this group one has also three kinds of nontrivial orbits for this case and could be calculated very easily. One can find it in for example [5]:

$$\begin{aligned}
O(0, 0) &= \{(0, 0)\}, \\
O(a, 0) &= \{(a, 0), (-a, a\tau), (a\tau, -a\tau), (-a\tau, a), (0, -a)\}, \\
O(0, b) &= \{(0, b), (b\tau, -b), (-b\tau, b\tau), (b, -b\tau), (-b, 0)\},
\end{aligned}$$

$$\begin{aligned} O(a, b) = \{(a, b), (-a, b + a\tau), (a\tau + b\tau, -b - a\tau), \\ (-a\tau - b\tau, a + b\tau), (b, -a - b\tau), (a + b\tau, -b), \\ (-a - b\tau, a\tau + b\tau), (b + a\tau, -a\tau - b\tau), (-b - a\tau, a), (-b, -a)\}. \end{aligned}$$

As before product of two orbits of  $H_2$  is written in terms of  $\lambda = (a_1, a_2) = a_1\omega_1 + a_2\omega_2$  and  $\lambda' = (b_1, b_2) = b_1\omega_1 + b_2\omega_2$  and simple roots, i.e.:

$$\begin{aligned} \langle \lambda | \alpha_j \rangle &= a_j, & \text{for } j = 1, 2 \text{ and } a_1, a_2, b_1, b_2 \in \mathbb{Z}^{\geq 0}. \\ \langle \lambda' | \alpha_j \rangle &= b_j, \end{aligned}$$

**Proposition 6.1.**

Decomposition of the product (4) of two orbits of  $H_2$  with dominant weights  $\lambda = (a_1, a_2)$  and  $\lambda' = (b_1, b_2)$  for  $a_1, a_2, b_1, b_2 \in \mathbb{Z}^{\geq 0}$  is given by the following formula

$$\begin{aligned} &O(\lambda) \otimes O(\lambda') && (17) \\ &= k_1 O(\lambda + \lambda') \\ &\cup k_2 O(|\langle \lambda - \lambda' | \alpha_1 \rangle|, \langle \lambda + \lambda' | \frac{\tau}{2}\alpha_1 + \alpha_2 \rangle - \frac{\tau}{2}|\langle \lambda - \lambda' | \alpha_1 \rangle|) \\ &\cup k_3 O(\min\{\langle \lambda | \alpha_1 \rangle + \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle, |\langle \lambda - \lambda' | \alpha_1 \rangle + \tau\langle \lambda | \alpha_2 \rangle|\}, \\ &\quad \min\{\langle \lambda' | \alpha_2 \rangle + \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle, |\langle \lambda - \lambda' | \alpha_2 \rangle - \tau\langle \lambda' | \alpha_1 \rangle|\}) \\ &\cup k_4 O(\min\{|\langle \lambda | \alpha_1 \rangle - \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle|, |\langle \lambda' | \alpha_1 \rangle - \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle|\}, \\ &\quad \min\{\langle \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 + \tau\alpha_2 \rangle, \langle \lambda' | \alpha_2 \rangle + \langle \lambda | \alpha_1 + \tau\alpha_2 \rangle, \\ &\quad |\langle \lambda - \lambda' | \tau\alpha_1 + \alpha_2 \rangle|\}) \\ &\cup k_5 O(|\min\{\langle \lambda | \tau\alpha_1 + \alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle, \langle \lambda | \alpha_1 \rangle + \langle \lambda' | \alpha_2 \rangle, \\ &\quad |\langle \lambda | \alpha_1 + \tau\alpha_2 \rangle - \langle \lambda' | \alpha_1 + \alpha_2 \rangle|\}|, |\min\{\langle \lambda | \alpha_1 \rangle + \langle \lambda' | \alpha_2 \rangle, \\ &\quad \langle \lambda' | \alpha_1 + \tau\alpha_2 \rangle - \langle \lambda | \alpha_2 \rangle, |\langle \lambda' | \tau\alpha_1 + \alpha_2 \rangle - \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle|\}|) \\ &\cup k_6 O(|\langle \lambda + \lambda' | \alpha_1 + \frac{\tau}{2}\alpha_2 \rangle - \frac{\tau}{2}|\langle \lambda - \lambda' | \alpha_2 \rangle|, |\langle \lambda - \lambda' | \alpha_2 \rangle|) \\ &\cup k_7 O(|\min\{\langle \lambda' - \lambda | \alpha_1 \rangle + \tau\langle \lambda' | \alpha_2 \rangle, \langle \lambda' | \alpha_1 \rangle + \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle\}|, \\ &\quad |\min\{\langle \lambda | \alpha_2 \rangle + \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle, \langle \lambda - \lambda' | \alpha_2 \rangle + \tau\langle \lambda | \alpha_1 \rangle\}|) \\ &\cup k_8 O(\min\{\langle \lambda | \alpha_1\tau + \alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle, \langle \lambda | \alpha_1 \rangle + \langle \lambda' | \alpha_1\tau + \alpha_2 \rangle, \\ &\quad |\langle \lambda - \lambda' | \alpha_1 + \tau\alpha_2 \rangle|\}, \min\{|\langle \lambda' | \alpha_2 \rangle - \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle|, \\ &\quad |\langle \lambda | \alpha_2 \rangle - \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle|\}) \\ &\cup k_9 O(|\min\{-\langle \lambda | \alpha_1 \rangle + \langle \lambda' | \tau\alpha_1 + \alpha_2 \rangle, \langle \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle, \\ &\quad |\langle \lambda' | \alpha_1 + \tau\alpha_2 \rangle - \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle|\}|, \\ &\quad |\min\{\langle \lambda | \alpha_1 + \tau\alpha_2 \rangle - \langle \lambda' | \alpha_2 \rangle, \langle \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle, \\ &\quad |\langle \lambda | \tau\alpha_1 + \alpha_2 \rangle - \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle|\}|) \\ &\cup k_{10} O(A, B), \end{aligned}$$

where the multiplicities  $k_1, \dots, k_{10}$  are given in terms of orbits sizes by:

$$\begin{aligned} A &= \left| \tau \max\{|\langle \lambda | \alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle|, |\langle \lambda' | \alpha_2 \rangle - \langle \lambda | \alpha_1 \rangle|\} \right. \\ &\quad \cdot \left( 1 - \text{sign}(\max\{0, \langle \lambda | \alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle, \langle \lambda' | \alpha_2 \rangle - \langle \lambda | \alpha_1 \rangle\}) \right) \\ &\quad \left. + \min\{|\langle \lambda | \alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle|, |\langle \lambda' | \alpha_2 \rangle - \langle \lambda | \alpha_1 \rangle|, \tau|\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle|\} \right|, \\ B &= \left| \left( \langle \lambda | \tau\alpha_1 + (1 + \tau)\alpha_2 \rangle - \langle \lambda' | (1 + \tau)\alpha_1 + \tau\alpha_2 \rangle \right) \right. \\ &\quad \cdot \left( 1 - |\text{sign}(\min\{0, \tau\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle, \langle \lambda' | \tau\alpha_1 + \alpha_2 \rangle - \langle \lambda | \alpha_1 + \tau\alpha_2 \rangle\})| \right) \\ &\quad \left. + \left( 1 - |\text{sign}(\min\{0, \langle \lambda | \alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle, \langle \lambda' | \alpha_2 \rangle - \langle \lambda | \alpha_1 \rangle\})| \right) \right| \end{aligned}$$

$$\begin{aligned}
& \cdot \max\{\langle \lambda | \tau\alpha_1 + \alpha_2 \rangle - \langle \lambda' | \alpha_1 + \tau\alpha_2 \rangle, \langle \lambda' | \tau\alpha_1 + \alpha_2 \rangle - \langle \lambda | \alpha_1 + \tau\alpha_2 \rangle\} \\
& + \left( \langle \lambda | (1+\tau)\alpha_1 + \tau\alpha_2 \rangle - \langle \lambda' | \tau\alpha_1 + (1+\tau)\alpha_2 \rangle \right) \\
& \cdot \left( 1 - |\text{sign}(\min\{0, \langle \lambda | \tau\alpha_1 + \alpha_2 \rangle - \langle \lambda' | \alpha_1 + \tau\alpha_2 \rangle\})| \right) \\
& \cdot \text{sign}(\max\{0, \langle \lambda' | \tau\alpha_1 + \alpha_2 \rangle - \langle \lambda | \alpha_1 + \tau\alpha_2 \rangle\}) \text{sign}(\min\{0, \tau\langle \lambda - \lambda' | \alpha_1 + \alpha_2 \rangle\}) \\
& + \max\{\langle \lambda | \alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle, \langle \lambda' | \alpha_2 \rangle - \langle \lambda | \alpha_1 \rangle\} \\
& \cdot \text{sign}(\min\{0, \langle \lambda | \alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle, \langle \lambda' | \alpha_2 \rangle - \langle \lambda | \alpha_1 \rangle\})
\end{aligned}$$

and

$$\begin{aligned}
k_1 &= \frac{1}{10} \frac{|O(\lambda)||O(\lambda')|}{|O(\lambda + \lambda')|} \\
k_2 &= \frac{1}{10} \frac{|O(\lambda)||O(\lambda')|}{|O(|\langle \lambda - \lambda' | \alpha_1 \rangle, \langle \lambda + \lambda' | \frac{\tau}{2}\alpha_1 + \alpha_2 \rangle - \frac{\tau}{2}|\langle \lambda - \lambda' | \alpha_1 \rangle|)|} \\
k_3 &= \frac{1}{10} |O(\lambda)||O(\lambda')| / \left| O(\min\{\langle \lambda | \alpha_1 \rangle + \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle, |\langle \lambda - \lambda' | \alpha_1 \rangle + \tau\langle \lambda | \alpha_2 \rangle|\}) \right. \\
&\quad \left. \min\{\langle \lambda' | \alpha_2 \rangle + \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle, |\langle \lambda - \lambda' | \alpha_2 \rangle - \tau\langle \lambda' | \alpha_1 \rangle|\} \right| \\
k_4 &= \frac{1}{10} |O(\lambda)||O(\lambda')| / \left| O(\min\{|\langle \lambda | \alpha_1 \rangle - \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle|, |\langle \lambda' | \alpha_1 \rangle - \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle|\}) \right. \\
&\quad \left. \min\{|\langle \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 + \tau\alpha_2 \rangle, |\langle \lambda' | \alpha_2 \rangle + \langle \lambda | \alpha_1 + \tau\alpha_2 \rangle, |\langle \lambda - \lambda' | \tau\alpha_1 + \alpha_2 \rangle|\} \right| \\
k_5 &= \frac{1}{10} |O(\lambda)||O(\lambda')| / \left| O(|\min\{\langle \lambda | \tau\alpha_1 + \alpha_2 \rangle - \langle \lambda' | \alpha_1 \rangle, \langle \lambda | \alpha_1 \rangle + \langle \lambda' | \alpha_2 \rangle, \right. \\
&\quad \left. |\langle \lambda | \alpha_1 + \tau\alpha_2 \rangle - \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle|\}|, |\min\{\langle \lambda | \alpha_1 \rangle + \langle \lambda' | \alpha_2 \rangle, \right. \\
&\quad \left. \langle \lambda' | \alpha_1 + \tau\alpha_2 \rangle - \langle \lambda | \alpha_2 \rangle, |\langle \lambda' | \tau\alpha_1 + \alpha_2 \rangle - \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle|\}|) \right| \\
k_6 &= \frac{1}{10} \frac{|O(\lambda)||O(\lambda')|}{|O(|\langle \lambda + \lambda' | \alpha_1 + \frac{\tau}{2}\alpha_2 \rangle - \frac{\tau}{2}|\langle \lambda - \lambda' | \alpha_2 \rangle|), |\langle \lambda - \lambda' | \alpha_2 \rangle|)} \\
k_7 &= \frac{1}{10} |O(\lambda)||O(\lambda')| / \left| O(|\min\{\langle \lambda' - \lambda | \alpha_1 \rangle + \tau\langle \lambda' | \alpha_2 \rangle, \langle \lambda' | \alpha_1 \rangle + \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle|\}, \right. \\
&\quad \left. |\min\{\langle \lambda | \alpha_2 \rangle + \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle, \langle \lambda - \lambda' | \alpha_2 \rangle + \tau\langle \lambda | \alpha_1 \rangle|\}|) \right| \\
k_8 &= \frac{1}{10} |O(\lambda)||O(\lambda')| / \left| O(\min\{\langle \lambda | \alpha_1 \tau + \alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle, \langle \lambda | \alpha_1 \rangle + \langle \lambda' | \alpha_1 \tau + \alpha_2 \rangle, \right. \\
&\quad \left. |\langle \lambda - \lambda' | \alpha_1 + \tau\alpha_2 \rangle|\}, \min\{|\langle \lambda' | \alpha_2 \rangle - \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle|, \right. \\
&\quad \left. |\langle \lambda | \alpha_2 \rangle - \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle|\}) \right| \\
k_9 &= \frac{1}{10} |O(\lambda)||O(\lambda')| / \left| O(|\min\{-\langle \lambda | \alpha_1 \rangle + \langle \lambda' | \tau\alpha_1 + \alpha_2 \rangle, \langle \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle, \right. \right. \\
&\quad \left. \left. |\langle \lambda' | \alpha_1 + \tau\alpha_2 \rangle - \tau\langle \lambda | \alpha_1 + \alpha_2 \rangle|\}|, |\min\{\langle \lambda | \alpha_1 + \tau\alpha_2 \rangle - \langle \lambda' | \alpha_2 \rangle, \right. \right. \\
&\quad \left. \left. \langle \lambda | \alpha_2 \rangle + \langle \lambda' | \alpha_1 \rangle, |\langle \lambda | \tau\alpha_1 + \alpha_2 \rangle - \tau\langle \lambda' | \alpha_1 + \alpha_2 \rangle|\}|) \right| \\
k_{10} &= \frac{1}{10} \left| \frac{O(\lambda)||O(\lambda')|}{|O(A, B)|} \right|.
\end{aligned}$$

The prove of (17) is again done by writing of existing list of all special cases following from it. Then each case of the list is easily verified.

#### Example 4.

$$\begin{aligned}
O(1, 2) \otimes O(3, 1) &= O(4, 3) \cup O(2, 3 + \tau) \cup O(2 - \tau, 3\tau - 1) \cup O(\tau - 1, 1) \\
&\cup O(2 + \tau, 1 + \tau) \cup O(4 + \tau, 1) \cup O(2\tau - 3, 3\tau - 2) \cup O(2\tau - 2, 3\tau - 1) \quad (18) \\
&\cup O(3\tau - 3, 2\tau - 1) \cup 2O(1, 0).
\end{aligned}$$

Orbits and decomposition of product (18) are presented in figure 4. More advanced examples one could find in appendix D.

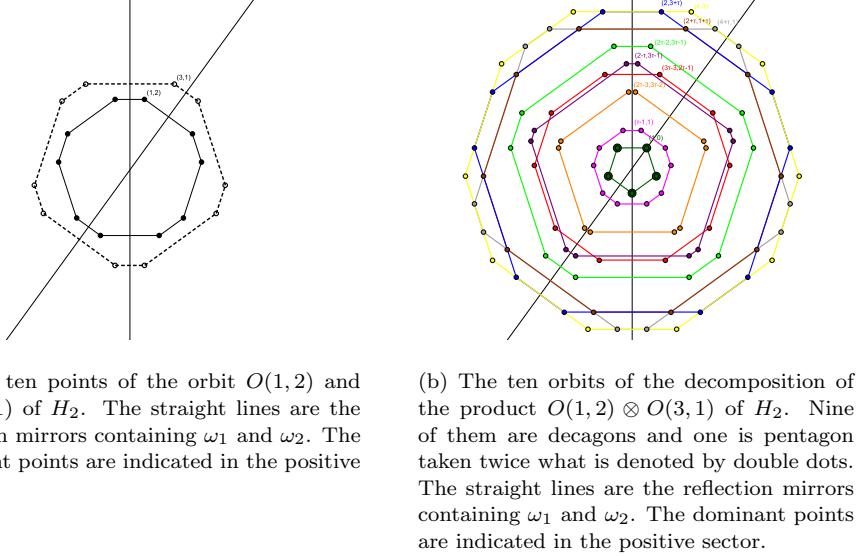


FIGURE 4.

## REMARKS

The orbits of Weyl groups of simple algebra are part of the weight system of irreducible representations. Working with the representations it can be advantageous to work with simple objects namely the Weyl group orbits rather than the weight systems. The group  $H_2$  is in the physics the dihedral group of order 10 and plays an important role in modeling two dimensional quasicrystals, see [22, 23, 24].

Orbits of the group are indispensable in defining families of orthogonal functions and polynomials [18, 25, 26]. The discretization of the orbit functions is interesting problem known for the Weyl groups [27] but is not known for  $H_2$  functions and polynomials, which are rather promising for exploration in digital data processing. The orthogonality of orbit functions of group  $H_2$  has not been found yet and it is challenging problem to be solved.

Similar problems for  $H_3$  and  $H_4$  groups could be solved by the same way.

The orbits could be viewed as polytopes. The decomposition of their product can be seen as an onion-like structure form by concentric orbits, [5, 13]

The formulas (7),(11),(14)(17) work also in the case when the orbits are points not belonging to the weight lattices. Their coordinates of their dominants weights are nonnegative real numbers. The congruence classes are lost in this case.

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## APPENDIX A

Sketch of proof of proposition 3.1.

*Proof.* Firstly when one rewrite (7) in terms of (6) then gets

$$\begin{aligned}
 O(a_1, a_2) \otimes O(b_1, b_2) &= k_1 O(a_1 + b_1, a_2 + b_2) & a_1, a_2, b_1, b_2 \in \mathbb{Z}^{\geq 0} \\
 &\cup k_2 O(|a_1 - b_1|, a_2 + b_2 + \min\{a_1, b_1\}) \\
 &\cup k_3 O(a_1 + b_1 + \min\{a_2, b_2\}, |a_2 - b_2|) \\
 &\cup k_4 O(|b_1 + \min\{a_2, b_2 - a_1\}|, |a_2 + \min\{b_1, a_1 - b_2\}|) \\
 &\cup k_5 O(|a_1 + \min\{b_2, a_2 - b_1\}|, |b_2 + \min\{a_1, b_1 - a_2\}|) \\
 &\cup k_6 O(|a_1 + a_2 - b_1 - b_2| - |\min\{a_1 - b_2, b_1 - a_2, 0\}|, \\
 &\quad ||a_1 + a_2 - b_1 - b_2| - |\min\{-a_1 + b_2, -b_1 + a_2, 0\}||),
 \end{aligned} \tag{19}$$

where

$$\begin{aligned}
 k_1 &= \frac{1}{6} \frac{|O(a_1, a_2)| |O(b_1, b_2)|}{|O(a_1 + b_1, a_2 + b_2)|} \\
 k_2 &= \frac{1}{6} \frac{|O(a_1, a_2)| |O(b_1, b_2)|}{|O(|a_1 - b_1|, a_2 + b_2 + \min\{a_1, b_1\})|} \\
 k_3 &= \frac{1}{6} \frac{|O(a_1, a_2)| |O(b_1, b_2)|}{|O(a_1 + b_1 + \min\{a_2, b_2\}, |a_2 - b_2|)|} \\
 k_4 &= \frac{1}{6} \frac{|O(a_1, a_2)| |O(b_1, b_2)|}{|O(|b_1 + \min\{a_2, b_2 - a_1\}|, |a_2 + \min\{b_1, a_1 - b_2\}|)|} \\
 k_5 &= \frac{1}{6} \frac{|O(a_1, a_2)| |O(b_1, b_2)|}{|O(|a_1 + \min\{b_2, a_2 - b_1\}|, |b_2 + \min\{a_1, b_1 - a_2\}|)|} \\
 k_6 &= \frac{1}{6} \frac{|O(a_1, a_2)| |O(b_1, b_2)|}{|O(|a_1 + a_2 - b_1 - b_2| - |\min\{a_1 - b_2, b_1 - a_2, 0\}|, ||a_1 + a_2 - b_1 - b_2| - |\min\{-a_1 + b_2, -b_1 + a_2, 0\}|)|}.
 \end{aligned}$$

It is obvious that (7) and (19) are equivalent. Now to check (19) it is enough to consider all special cases for  $a_1, a_2, b_1, b_2 \in \mathbb{Z}^{\geq 0}$ .

- First, let consider  $a_1 = a_2 = b_1 = b_2 = 0$ , then

$$\begin{aligned}
 O(0, 0) \otimes O(0, 0) &= \frac{1}{6} O(0, 0) \cup \frac{1}{6} O(0, 0) \\
 &= O(0, 0).
 \end{aligned}$$

Similarly, when  $a_1, a_2 \neq 0$  and  $b_1 = b_2 = 0$ , then

$$\begin{aligned}
 O(a_1, a_2) \otimes O(0, 0) &= \frac{1}{6} O(a_1, a_2) \cup \frac{1}{6} O(a_1, a_2) \cup \frac{1}{6} O(a_1, a_2) \cup \frac{1}{6} O(a_1, a_2) \cup \frac{1}{6} O(a_1, a_2) \\
 &= O(a_1, a_2).
 \end{aligned}$$

- Next special case which one gets from (7) or equivalently from (19) for  $0 \neq a_1 \neq b_1 \neq 0$  and  $a_2 = b_2 = 0$  following product

$$\begin{aligned}
 O(a_1, 0) \otimes O(b_1, 0) &= \frac{1}{2} O(a_1 + b_1, 0) \cup \frac{1}{4} O(|a_1 - b_1|, \min\{a_1, b_1\}) \cup \frac{1}{2} O(a_1 + b_1, 0) \\
 &\cup \frac{1}{4} O(|a_1 - b_1|, \min\{a_1, b_1\}) \cup \frac{1}{4} O(|a_1 - b_1|, \min\{a_1, b_1\}) \\
 &= O(a_1 + b_1, 0) \cup O(|a_1 - b_1|, \min\{a_1, b_1\}) \cup \frac{1}{4} O(|a_1 - b_1|, \min\{a_1, b_1\}) \\
 &= \begin{cases} O(a_1 - b_1, b_1) \cup O(a_1 + b_1, 0) & \text{for } a_1 > b_1 \\ O(-a_1 + b_1, a_1) \cup O(a_1 + b_1, 0) & \text{for } a_1 < b_1 \end{cases}.
 \end{aligned}$$

- When  $0 \neq a_1 = b_1 \neq 0$  and  $a_2 = b_2 = 0$ , then

$$\begin{aligned}
 O(a_1, 0) \otimes O(b_1, 0) &= \frac{1}{2} O(2a_1, 0) \cup \frac{1}{2} O(0, a_1) \cup \frac{1}{2} O(2a_1, 0) \cup \frac{1}{2} O(0, a_1) \\
 &\cup \frac{1}{2} O(0, a_1) \cup \frac{1}{2} O(0, a_1) = O(2a_1, 0) \cup 2O(0, a_1).
 \end{aligned}$$

- By analogy to above one can check that for  $a_1, b_2 \neq 0$  and  $a_2 = b_1 = 0$  the formula (7) takes the form

$$O(a_1, 0) \otimes O(0, b_2) = \begin{cases} O(a_1, b_2) \cup O(a_1 - b_2, 0) & \text{for } a_1 > b_2 \\ O(0, -a_1 + b_2) \cup O(a_1, b_2) & \text{for } a_1 < b_2 \\ O(a_1, a_1) \cup 3O(0, 0) & \text{for } a_1 = b_2 \end{cases}.$$

- The case  $O(0, a_2) \otimes O(0, b_2)$  is obtained by interchanging the first and second coordinates in each of orbits (in proofs of others propositions one has to uses symmetries for appropriate groups or check such case separately).
- When one considers  $a_1, a_2, b_1 \neq 0$  all different from each other and  $b_2 = 0$ , then the product  $O(a_1, a_2) \otimes O(b_1, 0)$  could be simplified to the form

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, 0) \\ = O(a_1 + b_1, a_2) \cup O(|a_1 + \min\{0, a_2 - b_1\}|, \min\{a_1, -a_2 + b_1\}) \\ \cup O(|a_1 - b_1|, a_2 + \min\{a_1, b_1\}). \end{aligned}$$

Now all special cases should be considered separately, i.e.:

- for  $a_2 > b_1, a_1 > b_1$  one gets

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, 0) = O(a_1 + b_1, a_2) \cup O(a_1 - b_1, a_2 + b_1) \\ \cup O(a_1, a_2 - b_1); \end{aligned}$$

- for  $a_2 > b_1, a_1 < b_1$  one gets

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, 0) = O(a_1 + b_1, a_2) \cup O(-a_1 + b_1, a_1 + a_2) \\ \cup O(a_1, a_2 - b_1); \end{aligned}$$

- for  $a_2 < b_1, a_1 > b_1$  one gets

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, 0) = O(a_1 + b_1, a_2) \cup O(a_1 - b_1, a_2 + b_1) \\ \cup O(a_1 + a_2 - b_1, -a_2 + b_1); \end{aligned}$$

- for  $a_2 < b_1, a_1 < b_1$  and  $a_1 + a_2 > b_1$  one gets

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, 0) = O(a_1 + b_1, a_2) \cup O(-a_1 + b_1, a_1 + a_2) \\ \cup O(a_1 + a_2 - b_1, -a_2 + b_1); \end{aligned}$$

- for  $a_2 < b_1, a_1 < b_1$  and  $a_1 + a_2 < b_1$  one gets

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, 0) = O(a_1 + b_1, a_2) \cup O(-a_1 + b_1, a_1 + a_2) \\ \cup O(-a_1 - a_2 + b_1, a_1). \end{aligned}$$

- In the case  $a_1 = b_1 \neq 0$  and  $b_2 = 0$  one gets

$$\begin{aligned} O(a_1, a_2) \otimes O(a_1, 0) &= O(2a_1, a_2) \cup O(\min\{a_1, a_2\}, |a_1 - a_2|) \\ &= \begin{cases} O(2a_1, a_2) \cup 2O(0, a_1 + a_2) \cup O(a_1, -a_1 + a_2) & \text{for } a_1 < a_2 \\ O(2a_1, a_2) \cup 2O(0, a_1 + a_2) \cup O(a_2, a_1 - a_2) & \text{for } a_1 > a_2 \end{cases}. \end{aligned}$$

- For other special case  $a_1 = a_2 = b_1 \neq 0$  and  $b_2 = 0$  one gets another subcases:

$$O(a_1, a_1) \otimes O(a_1, 0) = 2O(2a_1, a_2) \cup 2O(0, a_1 + a_2) \cup O(a_1, 0);$$

$$O(a_1, a_1) \otimes O(a_1 + a_2, 0) = 2O(0, a_1) \cup O(a_2, a_1 + a_2) \cup O(2a_1 + a_2, a_2);$$

$$O(a_1, a_1) \otimes O(2a_1, 0) = 2O(0, a_1) \cup O(a_1, 2a_1) \cup O(3a_1, a_1);$$

$$\begin{aligned} O(a_1, a_2) \otimes O(a_2, 0) \\ = 2O(a_1, 0) \cup O(a_1 + a_2, a_2) \cup O(|a_1 - a_2|, a_2 + \min\{a_1, a_2\}) \\ = \begin{cases} 2O(a_1, 0) \cup O(a_1 - a_2, 2a_2) \cup O(a_1 + a_2, a_2) & \text{for } a_1 > a_2 \\ 2O(a_1, 0) \cup O(-a_1 + a_2, a_1 + a_2) \cup O(a_1 + a_2, a_2) & \text{for } a_1 < a_2 \end{cases}. \end{aligned}$$

- Because of the biggest amount of subcases the most difficult to verify is the generic case, when  $a_1, a_2, b_1, b_2 \neq 0$ . Below some of them are presented:

◦ for  $-a_1 + b_2 > a_2 - b_1, a_1 < b_1$  and  $a_2 > b_2, a_1 + a_2 > b_1, a_1 > b_2$

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, b_2) = & O(a_1 + b_1, a_2 + b_2) \cup O(a_1 + a_2 - b_1, -a_2 + b_1 + b_2) \\ & \cup O(-a_1 + b_1, a_1 + a_2 + b_2) \cup O(-a_1 + b_1 + b_2, a_1 + a_2 - b_2) \\ & \cup O(a_1 + b_1 + b_2, a_2 - b_2) \cup O(-a_1 - a_2 + b_1 + b_2, a_1 - b_2); \end{aligned}$$

◦ for  $-a_1 + b_2 > a_2 - b_1$  and  $a_1 < b_1, a_2 > b_2$  and  $a_1 + a_2 > b_1$  and  $a_1 < b_2$

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, b_2) = & O(a_1 + b_1, a_2 + b_2) \cup O(a_1 + a_2 - b_1, -a_2 + b_1 + b_2) \\ & \cup O(-a_1 + b_1, a_1 + a_2 + b_2) \cup O(-a_1 + b_1 + b_2, a_1 + a_2 - b_2) \\ & \cup O(a_1 + b_1 + b_2, a_2 - b_2) \cup O(|a_2 - b_1|, -a_1 - a_2 + b_1 + b_2 + \min\{0, a_2 - b_1\}); \end{aligned}$$

◦ for  $-a_1 + b_2 > a_2 - b_1, a_1 < b_1$  and  $a_2 > b_2$  and  $a_1 + a_2 < b_1$

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, b_2) = & O(a_1 + b_1, a_2 + b_2) \cup O(-a_1 + b_1, a_1 + a_2 + b_2) \\ & \cup O(-a_1 - a_2 + b_1, a_1 + b_2) \cup O(-a_1 + b_1 + b_2, a_1 + a_2 - b_2) \\ & \cup O(a_1 + b_1 + b_2, a_2 - b_2) \cup O(|a_1 + a_2 - b_1 - b_2 + |\min\{0, a_1 - b_2\}|, |a_1 - b_2|); \end{aligned}$$

◦ for  $-a_1 + b_2 > a_2 - b_1$  and  $a_1 < b_1, a_2 < b_2$

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, b_2) = & O(a_1 + b_1, a_2 + b_2) \cup O(-a_1 + b_1, a_1 + a_2 + b_2) \\ & \cup O(a_1 + a_2 + b_1, -a_2 + b_2) \cup O(b_1 + \min\{a_2, -a_1 + b_2\}, |a_1 + a_2 - b_2|) \\ & \cup O(|a_1 + a_2 - b_1|, b_2 + \min\{a_1, -a_2 + b_1\}) \\ & \cup O(|a_1 + a_2 - b_1 - b_2 + |\min\{0, a_1 - b_2\}|, |a_1 + a_2 - b_1 - b_2 + |\min\{0, a_2 - b_1\}|); \end{aligned}$$

◦ for  $-a_1 + b_2 > a_2 - b_1, a_1 > b_1$

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, b_2) = & O(a_1 + b_1, a_2 + b_2) \cup O(a_1 - b_1, a_2 + b_1 + b_2) \\ & \cup O(a_1 + a_2 - b_1, -a_2 + b_1 + b_2) \cup O(a_1 + a_2 + b_1, -a_2 + b_2) \\ & \cup O(b_1 + \min\{a_2, -a_1 + b_2\}, |a_1 + a_2 - b_2|) \\ & \cup O(|a_1 + a_2 - b_1 - b_2 + |\min\{0, a_1 - b_2\}|, |(a_1 + a_2 - b_1 - b_2 + |\min\{0, a_2 - b_1\}|)|); \end{aligned}$$

◦ for  $-a_1 + b_2 < a_2 - b_1, a_2 > b_1$

$$\begin{aligned} O(a_1, a_2) \otimes O(b_1, b_2) = & O(a_1 + b_1, a_2 + b_2) \\ & \cup O(|a_1 - b_1|, a_2 + b_2 + \min\{a_1, b_1\}) \cup O(a_1 + b_1 + \min\{a_2, b_2\}, |a_2 - b_2|) \\ & \cup O(|-a_1 + b_1 + b_2|, a_2 + \min\{b_1, a_1 - b_2\}) \\ & \cup O(a_1 + \min\{a_2 - b_1, b_2\}, |-a_2 + b_1 + b_2|) \\ & \cup O(|-a_1 + b_2|, |-a_1 - a_2 + b_1 + b_2 + |\min\{0, -a_1 + b_2\}|)). \end{aligned}$$

- In the end one could present some special cases

$$O(a_1, a_1) \otimes O(a_1, a_1) =$$

$$O(2a_1, 2a_1) \cup 2O(0, 3a_1) \cup 2O(a_1, a_1) \cup 2O(3a_1, 0) \cup 6O(0, 0);$$

$$O(a_1, a_2) \otimes O(a_2, a_1) = O(a_1 + a_2, a_1 + a_2)$$

$$\cup O(a_1, a_1) \cup O(a_2, a_2) \cup O(|a_1 - a_2|, a_1 + a_2 + \min\{a_1, a_2\})$$

$$\cup O(a_1 + a_2 + \min\{a_1, a_2\}, |a_1 - a_2|) \cup 6O(0, 0).$$

Some cases could be obtained from others thanks to formula (4). One can check the rest of degenerated cases which are not described to check formulae to the end.

Each of the special cases of the formulae (7) equivalently (19) is easily verified directly. Hence the proof is completed.  $\square$

## APPENDIX B

One can prove proposition 4.1 by analogy to sketch of prove of proposition 3.1 presented in appendix A.

Calculation of some product of orbits of  $C_2$  is specially effective when one uses computer programs as Mathematica or Maple for such calculations using functions as min, sign and absolute value.

Below are presented some special cases for  $a, b > 0$  and  $a \neq b$ :

$$\begin{aligned} O(a, a) \otimes O(a, a) &= O(2a, 2a) \cup 2O(2a, a) \cup 2O(0, a) \cup 2O(0, 3a) \\ &\quad \cup 2O(2a, 0) \cup 2O(4a, 0) \cup 8O(0, 0); \\ O(a, b) \otimes O(a, b) &= O(2a, 2b) \cup 2O(0, a) \cup 2O(0, a + 2b) \cup 2O(2b, a) \\ &\quad \cup 2O(2a + 2b, 0) \cup 2O(2b, 0) \cup 8O(0, 0); \\ O(a, 0) \otimes O(b, 0) &= O(a + b, 0) \cup O(|a - b|, 0) \cup O(|a - b|, \min\{a, b\}); \\ O(a, b) \otimes O(b, a) &= O(a + b, a + b) \cup O(a + b, a) \cup O(a + b, b) \cup O(|a - b|, \min\{a, b\}) \\ &\quad \cup O(|a - b|, a + b + \min\{a, b\}) \cup O(a + b + \min\{a, b\}, |a - b|) \\ &\quad \cup O(a + b + 2\min\{a, b\}, |a - b|) \cup 2O(|a - b|, 0) \cup 2O(a + b, 0). \end{aligned}$$

## APPENDIX C

One can prove proposition 5.1 by analogy to sketch of prove of proposition 3.1 presented in appendix A.

Then using formula (14) one could present special cases:

$$\begin{aligned} O(a, a) \otimes O(a, a) &= O(2a, 2a) \cup 2O(a, 3a) \cup 2O(a, a) \cup 2O(0, 5a) \\ &\quad \cup 2O(0, 4a) \cup 2O(3a, 0) \cup 2O(2a, 0) \cup 2O(0, a) \cup 2O(a, 0) \cup 12O(0, 0); \\ O(a, 0) \otimes O(a, 0) &= O(2a, 0) \cup 2O(a, 0) \cup 2O(0, 3a) \cup 6O(0, 0); \\ O(a, b) \otimes O(a, b) &= O(2a, 2b) \cup 2O(0, b) \cup 2O(0, 3a + b) \cup 2O(0, 3a + 2b) \\ &\quad \cup 2O(a, 0) \cup 2O(a, b) \cup 2O(b, 3a) \cup 2O(a + b, 0) \\ &\quad \cup 2O(2a + b, 0) \cup 12O(0, 0); \\ O(a, b) \otimes O(b, a) &= O(a + b, a + b) \cup O(a, a + 2b) \cup O(b, 2a + b) \\ &\quad \cup O(|a - b|, a + b + \min\{3a, 3b\}) \cup O(\min\{a, 2b\}, |a - 2b|) \\ &\quad \cup O(a + b + \min\{a, b\}, |a - b|) \cup O(\min\{a, b\}, |a - b|) \\ &\quad \cup O(\min\{2a, b\}, |b - 2a|) \cup 2O(0, 2a + 2b) \cup 2O(0, |a - b|). \end{aligned}$$

## APPENDIX D

Non-crystallographic group  $H_2$  differ from crystallographic ones which are described in many papers and books in details in the opposite to the first one. Below some facts concerning  $H_2$ , collected from [1, 5, 19, 20, 22, 23], are presented.

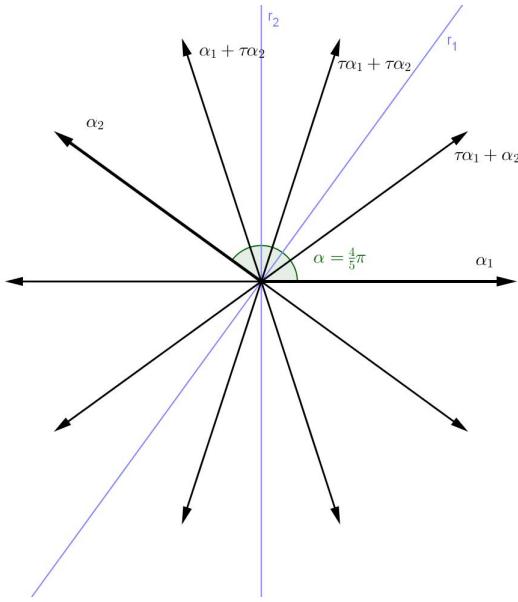
In the complex plane root system for  $H_2$  could be a set of 10th roots of unity

$$\left\{ \pm \zeta^j \mid \zeta = e^{\frac{2\pi i}{5}} \right\} \subset \mathbb{C}.$$

In this paper the simple roots were chosen as  $\alpha_1 = \sqrt{2}\zeta$ ,  $\alpha_2 = \sqrt{2}\zeta^2$ , then the highest root is  $\xi = \tau\alpha_1 + \tau\alpha_2$  (see figure 5) and

$$\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \tau\alpha_2), \pm(\tau\alpha_1 + \alpha_2), \pm\tau(\alpha_1 + \alpha_2)\},$$

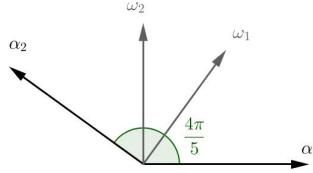
where  $\tau = \frac{1}{2}(1 + \sqrt{5}) = 2\cos\frac{\pi}{5}$ ,  $\tau' = \frac{1}{2}(1 - \sqrt{5}) = -2\cos\frac{2\pi}{5}$ . There are solutions of the equation on golden ratio  $x^2 - x - 1 = 0$ . It is easy to check that  $\tau \cdot \tau' = -1$ ,  $\tau + \tau' = 1$ .

FIGURE 5. The roots of group  $H_2$  and reflections  $r_1, r_2$ 

The Cartan matrix of  $H_2$  and inverse Cartan matrix are:

$$C = \begin{pmatrix} 2\langle \alpha_i | \alpha_j \rangle \\ \langle \alpha_j | \alpha_j \rangle \end{pmatrix} = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}, \quad C^{-1} = \frac{1}{5} \begin{pmatrix} 4+2\tau & 1+3\tau \\ 1+3\tau & 4+2\tau \end{pmatrix}.$$

The relation between the dual basis is given in terms of Cartan matrix, see (2) and it is presented in figure 6.

FIGURE 6. Dual basis to  $\alpha$  and  $\beta$ 

After presentation of basic facts for the group  $H_2$  one could calculate some special cases of product of two orbits for  $a, b > 0$  and  $a \neq b$

$$\begin{aligned} O(a, a) \otimes O(a, a) &= O(2a, 2a) \cup 2O(a\tau, a\tau) \cup 2O(-a + a\tau, -a + a\tau) \\ &\cup 2O(0, -a + 2a\tau) \cup 2O(-a + 2a\tau, 0) \\ &\cup 2O(2a + a\tau, 0) \cup 2O(0, 2a + a\tau) \cup 10O(0, 0); \\ O(a, 0) \otimes O(a, 0) &= O(2a, 0) \cup 2O(0, a\tau) \cup 2O(-a + a\tau, 0); \\ O(a, 0) \otimes O(0, a) &= O(a, a) \cup O(-a + a\tau, -a + a\tau) \cup 5O(0, 0); \end{aligned}$$

$$\begin{aligned}
O(a, b) \otimes O(a, b) &= O(2a, 2b) \cup 2O(b\tau, a\tau) \cup 2O(-a + a\tau, -b + b\tau) \\
&\cup 2O(0, 2b + a\tau) \cup 2O(2a + b\tau, 0) \\
&\cup 2O(0, -b + \tau(a + b)) \cup 2O(-a + \tau(a + b), 0) \\
&\cup 2O(\max\{0, a - b\}(-1 + \tau), \max\{0, b - a\}(-1 + \tau)) ; \\
O(a, b) \otimes O(b, a) &= O(a + b, a + b) \cup O(b - a + a\tau, b - a + a\tau) \cup O(a - b + b\tau, a - b + b\tau) \\
&\cup O(|a - b|, a + b + \min\{a, b\}\tau) \cup O(a + b + \min\{a, b\}\tau, |a - b|) \\
&\cup O(\tau \min\{a, b\} + (-1 + \tau) \max\{a, b\}, \tau|a - b|) \\
&\cup O(\tau|a - b|, \tau \min\{a, b\} + (-1 + \tau) \max\{a, b\}) \\
&\cup O(-b + b\tau, -b + b\tau) \cup O(-a + a\tau, -a + a\tau) \cup 10O(0, 0) .
\end{aligned}$$

Orbits products for  $H_2$  are the most interesting. Because of properties of  $\tau$  it could be rewrite in very different form, for example

$$O(a, a) \otimes O(a, a) = O(2a, 0) \cup 2O(0, a\tau) \cup 2O(\frac{a}{\tau}, 0)$$

and also one can calculate orbit products for not necessary integer numbers, for example for multiplication of  $\tau$  or  $-\tau' = -1 + \tau = \frac{1}{\tau}$

$$\begin{aligned}
O(\tau, 0) \otimes O(\tau, 0) &= O(2\tau, 0) \cup 2O(0, 1 + \tau) \cup 2O(1, 0) ; \\
O(\frac{1}{\tau}, 0) \otimes O(\frac{1}{\tau}, 0) &= O(-1 + \tau, 0) \otimes O(-1 + \tau, 0) = O(-\tau', 0) \otimes O(-\tau', 0) \\
&= O(\frac{2}{\tau}, 0) \cup 2O(\frac{1}{1+\tau}, 0) \cup 2O(0, 1) \\
&= O(-2 + 2\tau, 0) \cup 2O(2 - \tau, 0) \cup 2O(0, 1) \\
&= O(-2\tau', 0) \cup 2O(1 + \tau', 0) \cup 2O(0, 1) .
\end{aligned}$$

This group has the biggest potential from all groups described in this paper and it is very good staring point to another paper.

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